Stability Analysis for Spatial Attrition with n Forces

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Abstract. Close combat between two forces can be modelled through a set of two coupled partial differential equations, of second order in space and first order in time. That problem has been studied and shows a way to find stable solutions by means of a careful selection of the discretization both in time and space and through the use of a simple transformation. The results are generalized here for more interacting forces. It is found that the eigenvalues of the matrix that represent the system dynamics together with the time step size shape up the stability coefficients.

Keywords: PDE, stability, attrition, Lanchester, Spatio-temporal evolution
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INTRODUCTION

When modelling the interaction of two forces subject to diffusion and attrition or regeneration with a background velocity and assuming linear behavior, a set of coupled linear partial differential equations, with explicit intervention of Laplacians, first order temporal derivatives and gradients projected over the backround velocity vectors appears. The Finite Difference (FD) Method can be applied to solve this problem, because the main effects are likely to happen allover the dominion and because transient behavior is of particular interest. As confidence in the results is needed, it is paramount to analyze stability and confirm that both the stepping or the coarseness of the spatial grid is good enough. That analysis provided here offers a criterion for examining the formulation and allow the managing of the parameters, along with a transformation and de-transformation of the functions. This document is an extension of the work done by the authors in [1] and is intended to serve as a first approach to more complex problems like those found in mathematical biology [2, 3], ecology [4, 5, 6] and warfare [7, 8, 9].

THE MODEL

Interactions among n forces

Even though it is hard to find a system where every force fight against the other forces, the underlying assumption is that there are some mainstreams in the conflict but with some heterogeinity among them. One simple situation could include two main armies, one with $B1$ and $B2$ forces, and another one with $R1$ and $R2$ forces. Another very interesting situation is the simplified case with one mainstream with only one kind of troops R and the other mainstream consisting of three armies: B, $B1$ and $B2$. This case is intended to exploit the concept of sinergy, so as to have $B1$ and $B2$ that have some performance when they are spatially separated, but once they mix, they become a third force B with enhanced performance, better than that of $B1$ and $B2$. The general formulation suits all cases where four forces are interacting.

For the problem at stake, we use for illustrative purposes the case $n = 4$. The forces will be called blue, red, green and yellow (B, R, G and Y). Following the same diffusion behavior as in [1], the problem is expressed as:

$$\begin{bmatrix} p_B h_{BB}\nabla^2 - M_B - \vec{v}_{0B}\cdot\vec{\nabla} & -p_B h_{BR}\nabla^2 - E_{BR} & -p_B h_{BG}\nabla^2 - E_{BG} & -p_B h_{BY}\nabla^2 - E_{BY} \\ -p_R h_{RB}\nabla^2 - E_{RB} & p_R h_{RR}\nabla^2 - M_R - \vec{v}_{0R}\cdot\vec{\nabla} & -p_R h_{RG}\nabla^2 - E_{RG} & -p_R h_{RY}\nabla^2 - E_{RY} \\ -p_G h_{GB}\nabla^2 - E_{GB} & -p_G h_{GR}\nabla^2 - E_{GR} & p_G h_{GG}\nabla^2 - M_G - \vec{v}_{0G}\cdot\vec{\nabla} & -p_G h_{GY}\nabla^2 - E_{GY} \\ -p_Y h_{YB}\nabla^2 - E_{YB} & -p_Y h_{YR}\nabla^2 - E_{YR} & -p_Y h_{YG}\nabla^2 - E_{YG} & p_Y h_{YY}\nabla^2 - M_Y - \vec{v}_{0Y}\cdot\vec{\nabla} \end{bmatrix} \begin{bmatrix} B \\ R \\ G \\ Y \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} B \\ R \\ G \\ Y \end{bmatrix} \quad (1)$$

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STABILITY

No Regeneration, No Reinforcements

This is the case when the attrition coefficients M_i and $E_{i,j}$ assume non negative values. For this situation the time stepping of the equations for the linear case leads to:

$$\left[\left(\frac{2}{\Delta t}I+W\right)-Q\nabla^2\right]U^{n+1}=\left[\left(\frac{2}{\Delta t}I-W\right)-Q\nabla^2\right]U^n \tag{2}$$

On the other side, the Laplacian operator under Crank-Nicolson becomes:

$$\nabla^2 U\sim\frac{1}{(\Delta x)^2}\left\{U_{i-1,j}+U_{i,j-1}+U_{i,j+1}+U_{i+1,j}-4U_{i,j}\right\} \tag{3}$$

Now assuming local modes solution, which employs *von Neumann stability analysis*, as in [10], and extended to a four-functions system, the problem remains stable if the amplitude of the local modes is kept bounded, because of the assumption of local modes $U^n_{j,l}=\begin{bmatrix}\xi\\ \eta\\ \varphi\\ \psi\end{bmatrix}\rho^n e^{i\left(jk_x+lk_y\right)\Delta x}$. This implies that the relative amplitude ρ, should comply $\rho<1$, given the fact that the dependence of the local modes relative amplitude is represented by the succesive integer powers of ρ. Bearing that in mind, it is possible to write:

$$\left\{\left(\frac{2}{\Delta t}I+W\right)-Q\nabla^2\right\}\begin{bmatrix}\rho^{n+1}\xi\\ \rho^{n+1}\eta\\ \rho^{n+1}\varphi\\ \rho^{n+1}\psi\end{bmatrix}e^{i(jk_x+lk_y)\Delta x}=\left\{\left(\frac{2}{\Delta t}I-W\right)-Q\nabla^2\right\}\begin{bmatrix}\rho^{n}\xi\\ \rho^{n}\eta\\ \rho^{n}\varphi\\ \rho^{n}\psi\end{bmatrix}e^{i(jk_x+lk_y)\Delta x} \tag{4}$$

But the Laplacian, applied to the exponential, with $\Delta x=\Delta y$ gives the following result:

$$\nabla^2\sim\frac{1}{(\Delta x)^2}\left\{e^{ik_x\Delta x}+e^{-ik_x\Delta x}+e^{ik_y\Delta x}+e^{-ik_y\Delta x}-4\right\} \tag{5}$$

$$\nabla^2\sim\frac{1}{(\Delta x)^2}\left\{2\cos k_x\Delta x+2\cos k_y\Delta x-4\right\}=\frac{\alpha}{(\Delta x)^2} \tag{6}$$

also:

$$\vec{v}_{0\theta}\cdot\vec{\nabla}\sim\frac{i}{\Delta x}\left(v_{0\theta x}\sin k_x\Delta x+v_{0\theta y}\sin k_y\Delta x\right)\doteq i\beta_\theta \tag{7}$$

so this last operator adds up only to the each M_θ parameter as an imaginary component (phase shifting in $\pi/2$), where θ can be either B, R, G or Y in the $n=4$ case, then:

$$\left\{\left(\frac{2}{\Delta t}I+W\right)-\frac{\alpha}{(\Delta x)^2}Q\right\}\begin{bmatrix}\xi\\ \eta\\ \varphi\\ \psi\end{bmatrix}\rho=\left\{\left(\frac{2}{\Delta t}I-W\right)+\frac{\alpha}{(\Delta x)^2}Q\right\}\begin{bmatrix}\xi\\ \eta\\ \varphi\\ \psi\end{bmatrix} \tag{8}$$

Calling $P_\theta=M_\theta+i\beta_\theta$, for $\theta=B,R,G$ or Y, the determinant of the involved matrix must be nil. Dividing by $(\rho+1)$, while rewriting the matrix, the expression is now:

$$\det\begin{bmatrix}\left(\frac{2}{\Delta t}\right)\left(\frac{\rho-1}{\rho+1}\right)+\left(P_B-\frac{\alpha Q_{BB}}{(\Delta x)^2}\right) & \left(E_{BR}-\frac{\alpha Q_{BR}}{(\Delta x)^2}\right) & \left(E_{BG}-\frac{\alpha Q_{BG}}{(\Delta x)^2}\right) & \left(E_{BY}-\frac{\alpha Q_{BY}}{(\Delta x)^2}\right)\\ \left(E_{RB}-\frac{\alpha Q_{RB}}{(\Delta x)^2}\right) & \left(\frac{2}{\Delta t}\right)\left(\frac{\rho-1}{\rho+1}\right)+\left(P_R-\frac{\alpha Q_{RR}}{(\Delta x)^2}\right) & \left(E_{RG}-\frac{\alpha Q_{RG}}{(\Delta x)^2}\right) & \left(E_{RY}-\frac{\alpha Q_{RY}}{(\Delta x)^2}\right)\\ \left(E_{GB}-\frac{\alpha Q_{GB}}{(\Delta x)^2}\right) & \left(E_{GR}-\frac{\alpha Q_{GR}}{(\Delta x)^2}\right) & \left(\frac{2}{\Delta t}\right)\left(\frac{\rho-1}{\rho+1}\right)+\left(P_G-\frac{\alpha Q_{GG}}{(\Delta x)^2}\right) & \left(E_{GY}-\frac{\alpha Q_{GY}}{(\Delta x)^2}\right)\\ \left(E_{YB}-\frac{\alpha Q_{YB}}{(\Delta x)^2}\right) & \left(E_{YR}-\frac{\alpha Q_{YR}}{(\Delta x)^2}\right) & \left(E_{YG}-\frac{\alpha Q_{YG}}{(\Delta x)^2}\right) & \left(\frac{2}{\Delta t}\right)\left(\frac{\rho-1}{\rho+1}\right)+\left(P_Y-\frac{\alpha Q_{YY}}{(\Delta x)^2}\right)\end{bmatrix}=0 \tag{9}$$

And the previous equation is equivalent to finding the eigenvalues of the matrix C_4, where the eigenvalues are $\lambda=\left(\frac{2}{\Delta t}\right)\left(\frac{1-\rho}{1+\rho}\right)$, so $\rho_i=\frac{1-\frac{\Delta t}{2}\lambda_i}{1+\frac{\Delta t}{2}\lambda_i}=\frac{2-\lambda_i\cdot\Delta t}{2+\lambda_i\cdot\Delta t}$ and

$$\begin{bmatrix} \left(P_B - \frac{\alpha Q_{BB}}{(\Delta x)^2}\right) & \left(E_{BR} - \frac{\alpha Q_{BR}}{(\Delta x)^2}\right) & \left(E_{BG} - \frac{\alpha Q_{BG}}{(\Delta x)^2}\right) & \left(E_{BY} - \frac{\alpha Q_{BY}}{(\Delta x)^2}\right) \\ \left(E_{RB} - \frac{\alpha Q_{RB}}{(\Delta x)^2}\right) & \left(P_R - \frac{\alpha Q_{RR}}{(\Delta x)^2}\right) & \left(E_{RG} - \frac{\alpha Q_{RG}}{(\Delta x)^2}\right) & \left(E_{RY} - \frac{\alpha Q_{RY}}{(\Delta x)^2}\right) \\ \left(E_{GB} - \frac{\alpha Q_{GB}}{(\Delta x)^2}\right) & \left(E_{GR} - \frac{\alpha Q_{GR}}{(\Delta x)^2}\right) & \left(P_G - \frac{\alpha Q_{GG}}{(\Delta x)^2}\right) & \left(E_{GY} - \frac{\alpha Q_{GY}}{(\Delta x)^2}\right) \\ \left(E_{YB} - \frac{\alpha Q_{YB}}{(\Delta x)^2}\right) & \left(E_{YR} - \frac{\alpha Q_{YR}}{(\Delta x)^2}\right) & \left(E_{YG} - \frac{\alpha Q_{YG}}{(\Delta x)^2}\right) & \left(P_Y - \frac{\alpha Q_{YY}}{(\Delta x)^2}\right) \end{bmatrix} \doteq C_4. \tag{10}$$

Naming the eigenvalues λ_i, with $i = 1 \ldots n$. (in the example $n = 4$), the n eigenvalues can be obtained by using a numerical package that needs to be run many times according to a reasonably fine combination of the phase parameters $\gamma_x = k_x \Delta x$ and $\gamma_y = k_y \Delta x$, that take values from 0 to 2π. It should be pointed out that k_x and k_y characterize the local modes.

Once the eigenvalues are obtained, the n values for ρ are calculated, so the next step would be to plot each of the $|\rho_i|$ and see if the $3D$ plot shows a value greater than 1. Instead of that, and in order to get more precision, it is advisable to plot $\bar{\rho}_i = 1 - |\rho_i|$ and check if the results are always positive.

In general with this model, C_n is independent on the time variable, and depends on the size of the grid.

Reinforcements and Self Regeneration of One or More Forces

Assuming that one of the forces has the biggest negative value for $M_{\theta_1 \theta_2}$, where θ can be either B, R, G or Y, and by taking equation (1) and replacing the concentrations for the following functions $\theta = \chi \exp(st)$, where (θ, χ) can be either (B, b), (R, r), (G, g) or $(Y, \tilde{y})$, the time derivative allows to replace the terms M_θ by $M_\theta + s$, which means that if the value chosen for s is greater than $-M_\theta$ for every θ, the equation keeps the same shape as (1) but replacing the old value of M_θ by the new value of $s + M_\theta$. It follows now, that the values of M_θ in the system which renders the equations unstable, are replaced for $s + M_\theta$ in the system, leading to a stable solution.

This formulation solves the problem for b, r, g and $\tilde{y}$ by using the same numerical formulation, so the actual solution is obtained by multiplying that vector by $\exp(st)$. Also, this method can help in reducing the density of the grid in $x - y$ or the stepping in t, even for non-regenerating relations between the functions. It is noticed that the spatial envelope of the solutions for B and R does not change, so for a given time of evolution, the distribution of forces b, r, g and $\tilde{y}$ are the same as for B,R, G and Y, except for the exponential factor.

AN EXAMPLE

Assuming the interaction of two alliances of two forces each, for $4{,}000$ time steps, one of the coefficients got the value $-9.0323 \cdot 10^{-5}$, showing the need to adapt the information. Then, time was discretized $400 \cdot 10^6$ times, but the minimum value for one of the coefficients was $-9.0323 \cdot 10^{-10}$. As there are reinforcements, if all the forces share the same domain, blue forces will eventually become red forces, and yellow forces will become green forces. The next procedure was to maintain $4{,}000$ time steps and use $s = 4$, resulting in nearly guaranteed stability because the minimum value for one coefficient was $9.6811 \cdot 10^{-6}$, see Fig. 1. If $s = 5$, instead, the minimum value is $3.4681 \cdot 10^{-4}$. The likelihood of having eigenvalues of multiplicity higher than 1 is very low, so in general, this procedure should lead to a stable solution. All the eigenvalues (40.000) resulted from a grid of (100×100) cells.

FUTURE WORK

Problems from diverse disciplines will be analyzed with the back of this procedure. Spatial and temporal convolution is another main subject to be analyzed in the near future. Also some nonlinearities in the spatial attrition equations need efforts to decide if the solutions are truly reliable.

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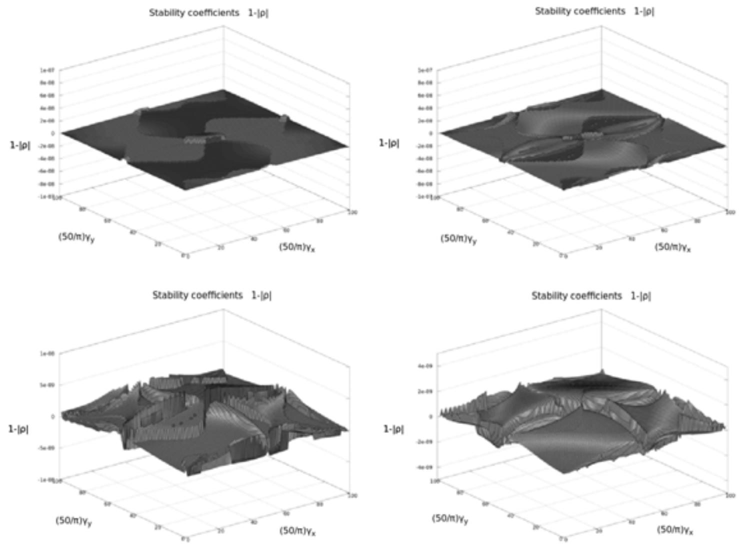

FIGURE 1. Stability coefficients for $s = 4$ and $4,000$ time steps.

TABLE 1. Main parameters

$$M = \begin{bmatrix} 3.9\cdot 10^{-7} & 2.1 & 1.3 & 1.4 \\ -2.1 & 4.0\cdot 10^{-7} & 1.75 & 1.2 \\ 2.5 & 1.8 & 4.2\cdot 10^{-7} & -1.5 \\ 1.3 & 2.3 & 1.5 & 4.3\cdot 10^{-7} \end{bmatrix}$$

$$Q = \begin{bmatrix} 4.4\cdot 10^{-7} & 5.0\cdot 10^{-7} & 5.0\cdot 10^{-7} & 4.4\cdot 10^{-7} \\ 5.2\cdot 10^{-7} & 5.3\cdot 10^{-7} & 5.0\cdot 10^{-7} & 5.0\cdot 10^{-7} \\ 5.1\cdot 10^{-7} & 6.0\cdot 10^{-7} & 5.5\cdot 10^{-7} & 5.0\cdot 10^{-7} \\ 5.0\cdot 10^{-7} & 5.0\cdot 10^{-7} & 4.7\cdot 10^{-7} & 5.1\cdot 10^{-7} \end{bmatrix}$$

$$\begin{aligned} \vec{v}_B &= 0.045\hat{i} + 0.020\hat{j} \\ \vec{v}_R &= -0.050\hat{i} - 0.030\hat{j} \\ \vec{v}_G &= 0.040\hat{i} + 0.025\hat{j} \\ \vec{v}_Y &= -0.030\hat{i} + 0.035\hat{j} \end{aligned}$$

Concept	Arbitrary Unit	Physical Unit
Time	UT	180 min
Distance	UL	$7.2\cdot 10^4$ m
Cell side	$7.368\cdot 10^{-4}UL$	53.05 m
Time step	$0.25\cdot 10^{-3}UT$	$45\cdot 10^{-3}$ min
Velocity	$UV = UL\cdot UT^{-1}$	400 m min^{-1}
Q	UL^2	see matrix
M	UT^{-1}	see matrix
Square of side	31 cells	
Time length	$0.1\,UT$	18 min

partial support

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